Real Analysis I **Exercise I** (due 09/25/2008)

(1) Show that the set of all sequences with values 0 or 1 is uncountable.

(2) Show that the set of real numbers is uncountable by proving the following:

(i) $(0,1) \approx \mathbf{R}$, that is, there is a one-to-one mapping from $\mathbf{R}$ onto $(0,1)$;

(ii) $(0,1)$ is uncountable.

(3) Let $\{x_n\}$ be a bounded sequence. Show that

$$\limsup(-x_n) = -\liminf x_n \quad \text{and} \quad \liminf(-x_n) = -\limsup x_n.$$

(4) If $\{x_n\}$ and $\{y_n\}$ are two bounded sequences, then show that

(i) $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$, and

(ii) $\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n$.

Moreover, show that if at least one of the sequences converges, the equality holds in (i) and (ii).

(5) Show that the sequence $\{x_n\}$ defined by $x_n = \left(1 + \frac{1}{n}\right)^n$ is a convergent sequence.

(6) Consider a sequence $\{(X_n, d_n)\}$ of metric spaces, and let $X = \Pi_{n=1}^{\infty} X_n$. For each $x = \{x_n\}$ and $y = \{y_n\}$ in X, define

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

(i) Show that d is a distance on X.

(ii) Show that (X, d) is a complete metric space if and only if each (X_n, d_n) is complete.

(iii) Show that (X, d) is compact if and only if each (X_n, d_n) is compact.

(7) Let $f : X \to X$ be a function from a set X into itself. A point $a \in X$ is called a *fixed point* for f if $f(a) = a$. Assume that (X, d) is a compact metric space and $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ for $x \neq y$. Show that f has a unique fixed point.

(8) Let (X, d) be a metric space. A function $f : X \to X$ is called a *contraction* if there exists some $0 < \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$; α is called a contraction constant. Show that every contraction f on a complete metric space (X, d) has a unique fixed point, that is, show that there exists a unique point $x \in X$ such that $f(x) = x$.

(9) Let (X, d) be a metric space. Define the distance of two nonempty subsets A and B of X by

$$d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}.$$

(i) Give an example of two closed sets A and B of some metric space such that $A \cap B = \emptyset$ and $d(A, B) = 0$.

(ii) If $A \cap B = \emptyset$, A is closed, and B is compact (and, of course, both are nonempty), then show that $d(A, B) > 0$.

(10) Show that $\lim_{n\to\infty} n^{1/n} = 1$.

(11) If $\{x_n\}$ is a sequence of strictly positive real numbers, then show that

$$\liminf \frac{x_{n+1}}{x_n} \leq \liminf x_n^{1/n} \leq \limsup x_n^{1/n} \leq \limsup \frac{x_{n+1}}{x_n}.$$

(12) If f is a continuous function on $[0, 1]$ such that $\int_0^1 x^n f(x)dx = 0$ for $n = 0, 1, 2, \cdots$, then show that $f(x) = 0$ for all $x \in [0, 1]$.

(13) Let $\{x_n\}$ be a sequence of real numbers. The number ℓ is called a *cluster point* of $\{x_n\}$ if given $\varepsilon > 0$ and given N, there exists $n \geq N$ such that $|x_n - \ell| < \varepsilon$. Show that $\limsup x_n$ and $\liminf x_n$ are the largest and smallest cluster points of the sequence $\{x_n\}$.

(14) If $\{x_n\}$ is a sequence of real numbers, then show that $\liminf x_n \leq \limsup x_n$ and that $\liminf x_n = \limsup x_n = \ell$ if and only if ℓ is the limit of $\{x_n\}$.

(15) Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers. Show that

$$\limsup x_n + \liminf y_n \leq \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n,$$

provided the right and left sides are not of the form $\infty - \infty$.

(16) Let $p > 1$ and $0 < x < 1$. Show that there is a sequence $\{x_n\}$ of integers with $0 \leq x_n < p$ such that

$$x = \sum_{n=1}^{\infty} \frac{x_n}{p^n}$$

and that this sequence is unique except when x is of the form q/p^n, in which case there are two such sequences. Show that, conversely, if $\{x_n\}$ is any sequence of integers with $0 \leq x_n < p$, then the series

$$\sum_{n=1}^{\infty} \frac{x_n}{p^n}$$

converges to a real number x with $0 \leq x \leq 1$. We note that for the case $p = 3$, this sequence is called the *ternary expansion* of x.

(17) The *Cantor set* C consists of all those real numbers in $[0, 1]$ that have ternary expansion (cf. Problem (16)) $\{x_n\}$ for which x_n is never 1. (If x has two ternary expansions, put x in C if one of the expansions has no term equal to 1.) Show that

(i) C is a closed set,

(ii) C can be put into a one-to-one correspondence with the interval $[0, 1]$, and

(iii) the set of accumulation points of C is the set itself.

(18) Let f be a real (or extended real) valued function defined for all x in an interval containing y. Then f is called *lower* (resp., *upper*) *semicontinuous* at y if $f(y) \neq -\infty$ and $f(y) \leq \liminf_{x \to y} f(x)$ (resp., $f(y) \neq +\infty$ and $f(y) \geq \limsup_{x \to y} f(x)$).

(i) If $f(y)$ is finite, show that f is lower semicontinuous at y if and only if given $\varepsilon > 0$, there exists $\delta > 0$ such that $f(y) \leq f(x) + \varepsilon$ for all x with $|x - y| < \delta$.

(ii) Show that f is continuous at y if and only if f is both upper and lower semicontinuous at y.

(iii) Show that a real-valued function f is lower semicontinuous on (a, b) if and only if the set $\{x : f(x) > \lambda\}$ is open for each real number λ.